

A Theory of Intermittency Differentiation of 1D Infinitely Divisible Multiplicative Chaos Measures

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Abstract

A theory of intermittency differentiation is developed for a general class of 1D Infinitely Divisible Multiplicative Chaos measures. The intermittency invariance of the underlying infinitely divisible field is established and utilized to derive a Feynman-Kac equation for the distribution of the total mass of the limit measure by considering a stochastic flow in intermittency. The resulting equation prescribes the rule of intermittency differentiation for a general functional of the total mass and determines the distribution of the total mass and its dependence structure to the first order in intermittency. For application, positive integer moments and covariance structure of the total mass are considered in detail.

1 Introduction

In this paper we contribute to the study of Infinitely Divisible Multiplicative Chaos (IDMC) measures, also known as limit log-infinitely divisible random measures, on the unit interval. This study was initiated by Mandelbrot [18], [19] and Bacry *et. al.* [2] in the limit lognormal case, extended to the compound Poisson case by Barral and Mandelbrot [5], and developed in the general infinitely divisible case by Bacry and Muzy [3], [20] based on the theory of Kahane [16], spectral representations of infinitely divisible processes of Rajput and Rosinski [34], and conical set constructions of Barral and Mandelbrot [5] and Schmitt and Marsan [35]. A different conical construction was introduced by Barral and Jin [4], who studied the problems of non-degeneracy of the limit measure and of the finiteness of its positive and negative integer moments in their model of infinitely divisible multiplicative chaos. We noted in [24] that the underlying infinitely divisible field of the Bacry-Muzy construction exhibits certain invariances and derived explicit multiple integral representations for single and joint positive integer moments of the total mass of the Bacry-Muzy measure in [30].

The interest in Bacry-Muzy multiplicative chaos measures stems from their properties of stochastic self-similarity with log-infinitely divisible multipliers, of being grid-free and stationary, and having exactly multiscaling moments. Due to these remarkable properties and the complexity of mathematical problems that they pose such as understanding the stochastic dependence structure of the limit measure, multiplicative chaos measures are generating a significant level of interest in mathematical physics, especially in the context of KPZ, cf. [6] and [32]. Of all such measures, the Gaussian Multiplicative Chaos (GMC) measure, also known as the limit lognormal measure in the literature, has gained the greatest amount of popularity as it appears in such diverse fields of mathematical physics as conformal field theory and quantum gravity [8], statistical mechanics of disordered energy landscapes and extrema of the 2D gaussian free field [10], [12], [13], [14], [17], [29], a theory of conformal weldings [1], and even in conjectured applications to the behavior of the Riemann zeta

function on the critical line [11], [28]. In addition, the moments of the total mass of the limit measure in 1D are known to be given by the Selberg integral on the interval, cf. [2], and by the Morris integral on the circle, cf. [10], hence connecting GMC theory with many areas of mathematics, where these integrals occur [9]. We refer the interested reader to [33] for a general review of the gaussian case, to [26] and [28] for detailed reviews of the Bacry-Muzy gaussian case on the interval and to [24] and [30] for the infinitely divisible case on the interval.

An important open problem in the theory of multiplicative chaos is to compute the distribution of the total mass of the limit measure and, more generally, understand its stochastic dependence structure. The primary challenge of this problem is that the underlying infinitely divisible field has strongly stochastically dependent increments so the usual Markovian techniques that work for Lévy processes do not apply. In addition, the limit measure is defined as a limit of the exponential functional of this field in the limit of zero scale, where the field diverges, so one has to deal with a singular limit of a strongly stochastically-dependent process. Moreover, the recovery of the total mass distribution from its moments is not possible in the sense of the classical moment problem as the moments diverge at any level of intermittency for most multiplicative chaos measures. To overcome these difficulties in the GMC case, we introduced in [21] a novel mathematical technique of intermittency differentiation that replaced time with intermittency and the non-existent Markov property of the underlying gaussian field with intermittency invariance. This technique allowed us to derive the rule of intermittency differentiation for a general class of functionals of the total mass in the form of a non-local Feynman-Kac equation. This equation in turn led to the intermittency renormalization solution to the moment problem and showed how to recover the distribution from the moments by systematically removing infinity from them, cf. [22]. Finally, in [23] we summed the intermittency (high temperature) expansion of the Mellin transform of the total mass in closed form and proved that the resulting formula is the Mellin transform of a valid positive probability distribution, which we termed the Selberg integral probability distribution, see [26], [27] for its detailed analytic and probabilistic analysis, respectively, which is then naturally conjectured to be that of the total mass. Interestingly, the same formula for the Mellin transform was obtained independently using a different technique in [12]. It is still an open problem to verify our conjecture, see [29] for a review of the Selberg integral probability distribution and the construction of the Morris integral probability distribution, which is conjectured, following the work of [10], to be the distribution of the total mass of the GMC measure on the circle. We refer the reader to [31] for a detailed discussion of these and other related conjectures.

The contribution of this paper is to extend the technique of intermittency differentiation to the general IDMC measure in 1D. Our setup, which we first introduced in [31] in the gaussian case, is slightly more general than that of Bacry-Muzy [3], [20] in that we keep their conical set construction but allow for a general intensity measure, subject to a positivity condition. We show that the intermittency invariance of the underlying infinitely divisible (ID) field, which we first established in [24] in the canonical Bacry-Muzy case, extends to the general framework of this paper. Our main contribution is the derivation of intermittency differentiation rules for both the total mass of the IDMC measure and its dependence structure in the form of non-local Feynman-Kac equations. These equations relate the intermittency derivative of a class of functionals of the total mass and the Lévy-Khinchine formula of the underlying ID distribution. We believe our results to be new for all multiplicative chaos measures other than the GMC and a first major step towards to the computation of the distribution of the total mass. In particular, they allow us to compute the intermittency derivative of the distribution at zero intermittency and its covariance structure explicitly.

Our derivation of the intermittency differentiation rule is exact in the sense of equality of formal

power series but not mathematically rigorous as we shun all questions of convergence. The rest of our results are rigorous. In particular, we give a rigorous derivation of the differentiation rule for positive integer moments of the total mass thereby verifying our main result in this case.

The structure of the paper is as follows. In Section 2 we review the Bacry-Muzy construction and extend it to a general intensity measure. In Section 3 we state our results. In Section 4 we give the proofs. Section 5 concludes.

2 Review and extension of the Bacry-Muzy construction

In this section we will review and somewhat extend the infinitely divisible multiplicative chaos (IDMC) construction of Bacry and Muzy [3] and [20], including the formula for positive integer moments of the total mass that we first noted in [30].

The starting point is an infinitely divisible (ID) independently scattered random measure P on the time-scale plane $\mathbb{H}_+ = \{(t, l), l > 0\}$, distributed uniformly with respect to some positive intensity measure ρ . This means that $P(A)$ is ID for measurable subsets $A \subset \mathbb{H}_+$, $P(A)$ and $P(B)$ are independent if $A \cap B = \emptyset$, and

$$\mathbf{E} \left[e^{iqP(A)} \right] = e^{\mu \phi(q) \rho(A)}, \quad q \in \mathbb{R}, \quad (2.1)$$

where $\mu > 0$ is the intermittency parameter¹ and $\phi(q)$ is the logarithm of the characteristic function of the underlying ID distribution and is given by the Lévy-Khinchine formula

$$\phi(q) = -\frac{iq\sigma^2}{2} - \frac{q^2\sigma^2}{2} + \int_{\mathbb{R} \setminus \{0\}} \left(e^{iqu} - 1 - iq(e^u - 1) \right) d\mathcal{M}(u). \quad (2.2)$$

It is normalized by $\phi(-i) = 0$ so that $\mathbf{E} \left[e^{P(A)} \right] = 1$ for all measurable subsets $A \subset \mathbb{H}_+$. The constant σ satisfies $\sigma^2 \geq 0$ and the spectral function $\mathcal{M}(u)$ is continuous and non-decreasing on $(-\infty, 0)$ and $(0, \infty)$, and satisfies the integrability and limit conditions $\int_{[-1, 1] \setminus \{0\}} u^2 d\mathcal{M}(u) < \infty$ and $\lim_{u \rightarrow \pm\infty} \mathcal{M}(u) = 0$. We will further assume that $\mathcal{M}(u)$ decays at infinity fast enough so that all integrals with respect to it in this and next sections converge, which restricts the class of permissible spectral functions. Next, following [5] and [35], Bacry and Muzy [20] introduce special conical sets $\mathcal{A}_\varepsilon(u)$ in the time-scale plane defined by

$$\mathcal{A}_\varepsilon(u) = \left\{ (t, l) \mid |t - u| \leq \frac{l}{2} \text{ for } \varepsilon \leq l \leq 1 \text{ and } |t - u| \leq \frac{1}{2} \text{ for } l \geq 1 \right\}. \quad (2.3)$$

The sets $\mathcal{A}_\varepsilon(u)$ and $\mathcal{A}_\varepsilon(v)$ intersect iff $|u - v| < 1$.

Our choice of the intensity measure will be somewhat more general than what was originally proposed by Bacry and Muzy [20]. Let

$$\rho(dt dl) = \frac{f(l)}{l^2} dt dl, \quad (2.4)$$

¹What we call μ and ρ is denoted by λ^2 and μ , respectively, in [20].

where the function $f(l)$ is defined by

$$\frac{f(l)}{l^2} = -\frac{d^2}{dl^2} \log r(l), \quad l \in (0, 1), \quad (2.5)$$

$$f(l) = \frac{d}{dz} \Big|_{z=1} \log r(z), \quad l \geq 1, \quad (2.6)$$

in terms of some function $r(t)$ that satisfies the properties

$$r(t) \text{ is positive, smooth and even on } (-1, 0) \cup (0, 1), \quad (2.7)$$

$$\lim_{t \rightarrow 0^+} t \frac{d}{dt} \log r(t) = 1, \quad (2.8)$$

and assume that $f(l)$ as defined in Eqs. (2.5) and (2.6) is positive. For example, the canonical choice of Bacry and Muzy corresponds to

$$r(t) = |t|, \quad (2.9)$$

$$f(l) = 1. \quad (2.10)$$

Let the function $\rho_\varepsilon(u, v)$ denote the intensity measure of intersections of the conical sets

$$\rho_\varepsilon(u, v) = \rho \left(\mathcal{A}_\varepsilon(u) \cap \mathcal{A}_\varepsilon(v) \right). \quad (2.11)$$

Clearly, $\rho_\varepsilon(u, v)$ is an even function of $u - v$ so that we can write $\rho_\varepsilon(u, v) = \rho_\varepsilon(|u - v|)$. It is easy to show, cf. the Appendix of [31], that it is given by

$$\rho_\varepsilon(u) = \begin{cases} -\log r(u) & \text{if } \varepsilon \leq |u| \leq 1, \\ -\log r(\varepsilon) + \left(1 - \frac{|u|}{\varepsilon}\right) \varepsilon \frac{d}{d\varepsilon} \log r(\varepsilon) & \text{if } |u| < \varepsilon, \end{cases} \quad (2.12)$$

and it is identically zero for $|u| > 1$. It is clear that $u \rightarrow P(\mathcal{A}_\varepsilon(u))$ is a stationary, ID process such that $P(\mathcal{A}_\varepsilon(u))$ and $P(\mathcal{A}_\varepsilon(v))$ are dependent iff $|u - v| < 1$. One can show that with probability one, the process $u \rightarrow P(\mathcal{A}_\varepsilon(u))$ has right-continuous trajectories with finite left limits.

Given these preliminaries, the IDMC measure $M_\mu(dt)$ on the interval $[0, 1]$ associated with $\phi(q)$ at intermittency μ is the zero scale limit $\varepsilon \rightarrow 0$ of finite scale random measures that are defined to be the exponential functional of the $u \rightarrow P(\mathcal{A}_\varepsilon(u))$ process.² To simplify notations, let

$$\omega_{\mu, \varepsilon}(u) \triangleq P(\mathcal{A}_\varepsilon(u)). \quad (2.13)$$

Then, the theorem of Bacry and Muzy states that the limit

$$M_\mu(a, b) = \lim_{\varepsilon \rightarrow 0} \int_a^b \exp(\omega_{\mu, \varepsilon}(u)) du, \quad (2.14)$$

exists in the weak a.s. sense. It was formally established in [3] based on the theory of Kahane [16] using the normalization of $\phi(q)$ and the property of P of being independently scattered. The limit measure has the stationarity property

$$M_\mu(t, t + \tau) \stackrel{\text{in law}}{=} M_\mu(0, \tau) \quad (2.15)$$

²We note in passing that one can symmetrize the conical set construction by considering conical sets on the torus as opposed to the upper-half plane, cf. [1] for details, and construct the limit measure on the circle in the same way.

and is non-degenerate in the sense of $\mathbf{E}[M_\mu(a, b)] = |b - a|$ under the assumption³ that

$$1 + i\mu\phi'(-i) = 1 - \mu \left(\frac{\sigma^2}{2} + \int_{\mathbb{R} \setminus \{0\}} (ue^u - e^u + 1) d\mathcal{M}(u) \right) > 0. \quad (2.16)$$

The moments $q > 1$ of $M_\mu(0, t)$ are finite under the following necessary and sufficient conditions

$$q - \mu\phi(-iq) > 1 \implies \mathbf{E}[M_\mu^q(0, t)] < \infty, \quad (2.17a)$$

$$\mathbf{E}[M_\mu^q(0, t)] < \infty \implies q - \mu\phi(-iq) \geq 1. \quad (2.17b)$$

The combination $q - \mu\phi(-iq)$ is known as the *multiscaling spectrum*, cf. Eq. (2.26) below. We have

$$q - \mu\phi(-iq) = q - \mu \left(\frac{\sigma^2}{2} (q^2 - q) + \int_{\mathbb{R} \setminus \{0\}} (e^{qu} - 1 - q(e^u - 1)) d\mathcal{M}(u) \right). \quad (2.18)$$

The best known analytical handle on the Bacry-Muzy construction is given in the following fundamental lemma due to [3], which we state in the slightly greater generality than the original as we allow for $f(l)$ as in Eqs. (2.5) and (2.6).

Lemma 2.1 (Main lemma) *Given $t_1 \leq \dots \leq t_n$ and q_1, \dots, q_n , the joint characteristic function of $\omega_{\mu, \varepsilon}(t_j)$, $j = 1 \dots n$, is*

$$\mathbf{E} \left[\exp \left(i \sum_{j=1}^n q_j \omega_{\mu, \varepsilon}(t_j) \right) \right] = \exp \left(\mu \sum_{p=1}^n \sum_{k=1}^p \alpha_{p,k} \rho_\varepsilon(t_p - t_k) \right), \quad (2.19)$$

where $\rho_\varepsilon(u)$ is defined in Eq. (2.12) and the coefficients $\alpha_{p,k}$ are given in terms of $\phi(q)$ by

$$\alpha_{p,k} = \phi(r_{k,p}) + \phi(r_{k+1,p-1}) - \phi(r_{k,p-1}) - \phi(r_{k+1,p}), \quad (2.20)$$

and $r_{k,p} = \sum_{m=k}^p q_m$ if $k \leq p$ and zero otherwise. In addition,

$$\sum_{p=1}^n \sum_{k=1}^p \alpha_{p,k} = \phi \left(\sum_{j=1}^n q_j \right). \quad (2.21)$$

The significance of this lemma cannot be overemphasized as it determines the positive integer moments of the limit measure and is the source of the intermittency invariance of the $\omega_{\mu, \varepsilon}(t)$ process as explained below.

A multiple integral representation of the positive integer moments of the total mass of the limit measure can be written down explicitly. Given $m \in \mathbb{N}$, define the quantity $d(m)$ by

$$d(m) \triangleq \sigma^2 + \int_{\mathbb{R} \setminus \{0\}} e^{(m-1)u} (e^u - 1)^2 d\mathcal{M}(u). \quad (2.22)$$

³The non-degeneracy condition given in [3] is less stringent than Eq. (2.16), which is however sufficient in most cases of interest such as those of the limit lognormal and Poisson measures.

Then, the n th moment of the total mass is given by a generalized Selberg integral of dimension n . Let $0 \leq a < b \leq 1$ and n satisfy Eq. (2.17a).

$$\mathbf{E} \left[\left(\int_a^b M_\mu(dt) \right)^n \right] = n! \int_{a < t_1 < \dots < t_n < b} \prod_{k < p}^n r(t_p - t_k)^{-\mu d(p-k)} dt^{(n)}. \quad (2.23)$$

This result is due to [2] in the canonical gaussian case ($d\mathcal{M}(u) = 0$, $r(t) = |t|$), to [31] for general $r(t)$ in the gaussian case, and to [30] in general. The proof that we gave in [30] was limited to $r(t) = |t|$, however, the argument goes through verbatim in general and is based on a simple application of Lemma 2.1 in the form of the identity

$$\exp \left(\sum_{k < p}^n \mu d(p-k) \rho_\varepsilon(t_p - t_k) \right) = \mathbf{E} \left[e^{\omega_{\mu,\varepsilon}(t_1) + \dots + \omega_{\mu,\varepsilon}(t_n)} \right] \quad (2.24)$$

for any $0 < t_1 < \dots < t_n < 1$, and Fubini's theorem. The coefficients $d(m)$ have the important property

$$\sum_{k < p}^n d(p-k) = \phi(-in). \quad (2.25)$$

This equation combined with $r(t) \sim t$ in the limit $t \rightarrow 0$, cf. Eq. (2.8), implies

$$\mathbf{E} \left[\left(\int_0^t M_\mu(ds) \right)^n \right] \sim \text{const } t^{n-\mu\phi(-in)}, \quad t \rightarrow 0, \quad (2.26)$$

hence the significance of $q - \mu\phi(-iq)$ as the multiscaling spectrum of the measure.

We note parenthetically that the formula in Eq. (2.23) suggests how to define the Selberg integral in the general ID case.

$$S_n(\lambda, \lambda_1, \lambda_2) \triangleq \int_{0 < t_1 < \dots < t_n < 1} \prod_{i=1}^n r(t_i)^{\lambda_1 d(i)} r(1-t_i)^{\lambda_2 d(n-i+1)} \prod_{k < p}^n r(t_p - t_k)^{2\lambda d(p-k)} dt^{(n)} \quad (2.27)$$

for generally complex λ , λ_1 , and λ_2 . Some of its properties in the general ID case and $r(t) = |t|$ are derived in [30].

We end this section with two main examples of IDMC measures: gaussian (GMC) and Poisson.

Limit lognormal measure Let $\sigma = 1$, $\mathcal{M}(u) = 0$ in Eq. (2.2), and $r(t) = |t|$. Then,

$$\mathbf{E} \left[\left(\int_0^1 M_\mu(dt) \right)^n \right] = n! \int_{0 < t_1 < \dots < t_n < 1} \prod_{k < p}^n |t_p - t_k|^{-\mu} dt^{(n)}. \quad (2.28)$$

Note that the nondegeneracy condition in Eq. (2.16) amounts to $0 < \mu < 2$ and that the moments become infinite for $n > 2/\mu$.

Limit Log-Poisson moments Let $\sigma = 0$ and $d\mathcal{M}(u) = \delta(u - \log(c))du$ in Eq. (2.2), i.e. the underlying distribution is a point mass at $\log(c)$, $c > 0$, $c \neq 1$.

$$\mathbf{E} \left[\left(\int_0^1 M_\mu(dt) \right)^n \right] = n! \int_{0 < t_1 < \dots < t_n < 1} \prod_{k < p}^n |t_p - t_k|^{-\mu(c-1)^2 c^{p-k-1}} dt^{(n)}. \quad (2.29)$$

The nondegeneracy condition in Eq. (2.16) is

$$0 < \mu < \frac{1}{c \log(c) - c + 1}, \quad (2.30)$$

so that the limit log-Poisson measure exists for any such c as $c \log(c) - c + 1 > 0$ for $c > 0$, $c \neq 1$. The moments are finite for $q > 1$ if

$$q - \mu(c^q - 1 - q(c - 1)) > 1 \implies \mathbf{E}[M_\mu(0, 1)^q] < \infty, \quad (2.31)$$

cf. Eqs. (2.17a) and (2.18). In particular, the moments become eventually infinite if $c > 1$ as they do in the limit lognormal case. On the contrary, if $0 < c < 1$, all moments for $q > 1$ are finite for sufficiently small μ .

3 Intermittency invariance and differentiation rule

In this section we will formulate the intermittency invariance of the underlying infinitely divisible (ID) field and state our main results on the rule of intermittency differentiation and its application to the distribution of the total mass at the lowest non-trivial order in intermittency. The proofs are deferred to Section 4.

Fix $L \geq 1$ and define the ID random variable by the formula

$$\mathbf{E}[e^{iqZ_L}] = e^{\mu\phi(q)\log L}. \quad (3.1)$$

Define the corresponding ID field by

$$\omega_{\mu,L,\varepsilon}(u) = \omega_{\mu,\varepsilon}(u) + Z_L, \quad (3.2)$$

where Z_L is independent of the process $\omega_{\mu,\varepsilon}(u)$. Clearly, $\omega_{\mu,L=1,\varepsilon}(u)$ coincides with the original field as defined in Eq. (2.13). Finally, let $\delta \rightarrow X(\delta)$ be a Lévy process (a stochastic process with stationary, independent increments) that is independent of the $u \rightarrow \omega_{\mu,L,\varepsilon}(u)$ process and defined in terms of the ID distribution associated with $\phi(q)$ as follows

$$\mathbf{E}[e^{iqX(\delta)}] = e^{\delta\phi(q)}, \quad X(0) = 0. \quad (3.3)$$

The existence and uniqueness of $X(\delta)$ follow from the general theory of Lévy processes, confer [7]. Then, we have the following result.

Theorem 3.1 (Intermittency invariance) *Fix μ , L , ε , and $\delta < \mu$, and $\bar{\omega}_{\delta,eL,\varepsilon}(t)$ denote an independent copy of the $\omega_{\mu,L,\varepsilon}(u)$ process with the intermittency δ and L replaced with eL , where e is the base of the natural logarithm. Then, there holds the following equality in law of stochastic processes in u on the interval $u \in [0, 1]$,*

$$X(\delta) + \omega_{\mu,L,\varepsilon}(u) = \omega_{\mu-\delta,L,\varepsilon}(u) + \bar{\omega}_{\delta,eL,\varepsilon}(u). \quad (3.4)$$

In the gaussian case this result is originally due to [21], [22] for $r(t) = |t|$ and to [31] for general $r(t)$. In the ID case this result is due to [24] for $r(t) = |t|$. Its extension to the general $r(t)$ is new.

The significance of the intermittency invariance is that it provides a technical device that replaces the non-existent Markov property of the underlying ID field and allows one to derive a Feynman-Kac equation for the distribution of the total mass by considering a stochastic flow in intermittency as opposed to time (as in the classical framework of diffusions).

Define the finite scale total mass to be

$$M_{\mu,\varepsilon} \triangleq \int_0^1 e^{\omega_{\mu,\varepsilon}(s)} ds \quad (3.5)$$

so that the total mass is the limit

$$M_\mu = \lim_{\varepsilon \rightarrow 0} M_{\mu,\varepsilon}. \quad (3.6)$$

Define also

$$g(s_1, s_2) \triangleq -\log r(s_1 - s_2), \quad (3.7)$$

which is the limit of the ρ measure of intersection of the conical sets, cf. Eq. (2.12). Then, given a smooth test function $F(x)$, our main results are as follows.

Theorem 3.2 (Rule of intermittency differentiation)

$$\begin{aligned} \frac{\partial}{\partial \mu} \mathbf{E}[F(M_\mu)] &= \sigma^2 \int_{\{s_1 < s_2\}} \lim_{\varepsilon \rightarrow 0} \mathbf{E} \left[F^{(2)}(M_{\mu,\varepsilon}) e^{\omega_{\mu,\varepsilon}(s_1) + \omega_{\mu,\varepsilon}(s_2)} \right] g(s_1, s_2) ds^{(2)} + \\ &+ \sum_{k=2}^{\infty} \int_{\mathbb{R} \setminus \{0\}} (e^u - 1)^k d\mathcal{M}(u) \int_{\{s_1 < \dots < s_k\}} \lim_{\varepsilon \rightarrow 0} \mathbf{E} \left[F^{(k)}(M_{\mu,\varepsilon}) e^{\omega_{\mu,\varepsilon}(s_1) + \dots + \omega_{\mu,\varepsilon}(s_k)} \right] g(s_1, s_k) ds^{(k)}. \end{aligned} \quad (3.8)$$

In the special case of the GMC this result appeared first in [21] and [22] for $r(t) = |t|$, and in [31] for general $r(t)$. In the ID case this result is new in all cases. A derivation of Theorem 3.2 from Theorem 3.1 is given in Section 4. It suffices to explain here that the main idea is to consider a stochastic flow in intermittency and evaluate the limit

$$\frac{\partial}{\partial \delta} \Big|_{\delta=0} \mathbf{E} \left[F(z e^{X(\delta)} M_{\mu,\varepsilon}) \right], \quad (3.9)$$

where $X(\delta)$ is defined in Eq. (3.3) and is independent of $\omega_{\mu,\varepsilon}(s)$, in two different ways: by the backward Kolmogorov equation for the Lévy processes $X(\delta)$ and by applying Theorem 3.1 and expanding to the first order in δ .

Upon substituting $\mu = 0$ into Eq. (3.8) we obtain an explicit formula for the first order term in the expansion of the distribution of the total mass in intermittency.

Corollary 3.3 (Distribution to the first order in intermittency) *The distribution of the total mass to the first order in intermittency is determined by*

$$\begin{aligned} \frac{\partial}{\partial \mu} \Big|_{\mu=0} \mathbf{E}[F(M_\mu)] &= \sigma^2 F^{(2)}(1) \int_{\{s_1 < s_2\}} g(s_1, s_2) ds^{(2)}, \\ &+ \sum_{k=2}^{\infty} F^{(k)}(1) \int_{\mathbb{R} \setminus \{0\}} (e^u - 1)^k d\mathcal{M}(u) \int_{\{s_1 < \dots < s_k\}} g(s_1, s_k) ds^{(k)}. \end{aligned} \quad (3.10)$$

Remark It is easy to show that

$$\int_{\{s_1 < \dots < s_k\}} g(s_1, s_k) ds^{(k)} \sim 1/k! \quad (3.11)$$

so that the sum in Eq. (3.10) is finite provided

$$\int_{\mathbb{R} \setminus \{0\}} F(e^u) d\mathcal{M}(u) < \infty. \quad (3.12)$$

The technique of intermittency differentiation is not limited to the distribution of the total mass of the limit measure but applies also to the joint distribution of the measure of subintervals, *i.e.* the dependence structure of the measure. We will illustrate this application with the case of two disjoint subintervals $I_1, I_2 \subset [0, 1]$, $I_1 \cap I_2 = \emptyset$, $\sup I_1 < \min I_2$ for simplicity, although it applies to any finite number of such subintervals. Denote

$$M_{\mu, \varepsilon}(I) \triangleq \int_I e^{\omega_{\mu, \varepsilon}(s)} ds \quad (3.13)$$

and the limit measure of the interval

$$M_\mu(I) = \lim_{\varepsilon \rightarrow 0} M_{\mu, \varepsilon}(I). \quad (3.14)$$

Theorem 3.4 (Intermittency differentiation for two intervals)

$$\begin{aligned} \frac{\partial}{\partial \mu} \mathbf{E}[F_1(M_\mu(I_1))F_2(M_\mu(I_2))] &= \sigma^2 \left[\int_{\{s_1 < s_2\} \subset I_1} \lim_{\varepsilon \rightarrow 0} \mathbf{E}[F_1^{(2)}(M_{\mu, \varepsilon}(I_1))F_2(M_{\mu, \varepsilon}(I_2))e^{\omega_{\mu, \varepsilon}(s_1) + \omega_{\mu, \varepsilon}(s_2)}] \times \right. \\ &\quad \times g(s_1, s_2) ds^{(2)} + \int_{\{s_1 < s_2\} \subset I_2} \lim_{\varepsilon \rightarrow 0} \mathbf{E}[F_1(M_{\mu, \varepsilon}(I_1))F_2^{(2)}(M_{\mu, \varepsilon}(I_2)) \times \\ &\quad \times e^{\omega_{\mu, \varepsilon}(s_1) + \omega_{\mu, \varepsilon}(s_2)}] g(s_1, s_2) ds^{(2)} + \int_{\{s_1 \in I_1, s_2 \in I_2\}} \lim_{\varepsilon \rightarrow 0} \mathbf{E}[F_1^{(1)}(M_{\mu, \varepsilon}(I_1)) \times \\ &\quad \times F_2^{(1)}(M_{\mu, \varepsilon}(I_2))e^{\omega_{\mu, \varepsilon}(s_1) + \omega_{\mu, \varepsilon}(s_2)}] g(s_1, s_2) ds^{(2)} \Big] + \\ &+ \sum_{\substack{k, l \geq 0 \\ k+l \geq 2}} \int_{\mathbb{R} \setminus \{0\}} (e^u - 1)^{k+l} d\mathcal{M}(u) \int_{\substack{\{s_1 < \dots < s_k\} \subset I_1 \\ \{s_{k+1} < \dots < s_{k+l}\} \subset I_2}} \lim_{\varepsilon \rightarrow 0} \mathbf{E}[F_1^{(k)}(M_{\mu, \varepsilon}(I_1)) \times \\ &\quad \times F_2^{(l)}(M_{\mu, \varepsilon}(I_2))e^{\omega_{\mu, \varepsilon}(s_1) + \dots + \omega_{\mu, \varepsilon}(s_{k+l})}] g(s_1, s_{k+l}) ds^{(k+l)}. \end{aligned} \quad (3.15)$$

This result in the gaussian case is originally due to [25]. In the infinitely divisible case it is new.

Corollary 3.5 (Joint distribution to the first order in intermittency)

$$\begin{aligned}
\left. \frac{\partial}{\partial \mu} \right|_{\mu=0} \mathbf{E}[F_1(M_\mu(I_1))F_2(M_\mu(I_2))] &= \sigma^2 \left[F_1^{(2)}(|I_1|)F_2(|I_2|) \int_{\{s_1 < s_2\} \subset I_1} g(s_1, s_2) ds^{(2)} + \right. \\
&+ F_1(|I_1|)F_2^{(2)}(|I_2|) \int_{\{s_1 < s_2\} \subset I_2} g(s_1, s_2) ds^{(2)} + \\
&+ F_1^{(1)}(|I_1|)F_2^{(1)}(|I_2|) \int_{\{s_1 \in I_1, s_2 \in I_2\}} g(s_1, s_2) ds^{(2)} \Big] + \\
&+ \sum_{\substack{k, l \geq 0 \\ k+l \geq 2}}^\infty F_1^{(k)}(|I_1|)F_2^{(l)}(|I_2|) \int_{\mathbb{R} \setminus \{0\}} (e^u - 1)^{k+l} d\mathcal{M}(u) \times \\
&\times \int_{\substack{\{s_1 < \dots < s_k\} \subset I_1 \\ \{s_{k+1} < \dots < s_{k+l}\} \subset I_2}} g(s_1, s_{k+l}) ds^{(k+l)}. \tag{3.16}
\end{aligned}$$

This result has an application to the problem of computing the covariance structure of the total mass distribution.

Corollary 3.6 (Covariance structure) *Let $0 < t < 1$. Then, in the limit $\tau \rightarrow 0$,*

$$\text{Cov}\left(\log \int_t^{t+\tau} M_\mu(dt), \log \int_0^\tau M_\mu(dt)\right) = \mu g(t) \left(\sigma^2 + \int_{\mathbb{R} \setminus \{0\}} u^2 d\mathcal{M}(u) \right) + O(\tau). \tag{3.17}$$

This result in the gaussian case and $r(t) = |t|$ was originally established in [2] and extended to the infinitely divisible case in [30]. Both calculations relied on a heuristic analytic continuation of joint integer moments. Corollary 3.5 allows us to dispense with the analytic continuation, as shown in Section 4.

We will illustrate the rule of intermittency differentiation with the example of positive integer moments. Recall the formula for the moments in Eq. (2.23), assuming n satisfies Eq. (2.17a). The intermittency derivative is then

$$\frac{\partial}{\partial \mu} \mathbf{E}[M_\mu^n] = n! \int_{0 < t_1 < \dots < t_n < 1} \sum_{k < p}^n d(p-k) g(t_p, t_k) \prod_{k < p}^n r(t_p - t_k)^{-\mu} d(p-k) dt^{(n)}. \tag{3.18}$$

On the other hand, given $F(x) = x^n$, Theorem 3.2 gives us the formula

$$\begin{aligned}
\frac{\partial}{\partial \mu} \mathbf{E}[M_\mu^n] &= \sigma^2 n(n-1) \int_{\{s_1 < s_2\}} \lim_{\varepsilon \rightarrow 0} \mathbf{E} \left[M_{\mu, \varepsilon}^{n-2} e^{\omega_{\mu, \varepsilon}(s_1) + \omega_{\mu, \varepsilon}(s_2)} \right] g(s_1, s_2) ds^{(2)} + \\
&+ \sum_{k=2}^n \int_{\mathbb{R} \setminus \{0\}} \frac{n!}{(n-k)!} (e^u - 1)^k d\mathcal{M}(u) \int_{\{s_1 < \dots < s_k\}} \lim_{\varepsilon \rightarrow 0} \mathbf{E} \left[M_{\mu, \varepsilon}^{n-k} e^{\omega_{\mu, \varepsilon}(s_1) + \dots + \omega_{\mu, \varepsilon}(s_k)} \right] g(s_1, s_k) ds^{(k)}. \tag{3.19}
\end{aligned}$$

The equivalence of Eqs. (3.18) and (3.19) is a corollary of the following general integral identity.

Theorem 3.7 (Intermittency derivative of integer moments) *Let $\omega(s)$ and $g(s, t)$ be continuous functions and $k = 2 \cdots n$. The identity*

$$\begin{aligned} \frac{1}{(n-k)!} \left(\int_0^1 e^{\omega(s)} ds \right)^{n-k} \int_{\{s_1 < \cdots < s_k\}} e^{\omega(s_1) + \cdots + \omega(s_k)} g(s_1, s_k) ds^{(k)} = \int_{\{s_1 < \cdots < s_n\}} e^{\omega(s_1) + \cdots + \omega(s_n)} \times \\ \times \left[\sum_{\substack{i < j \\ j-i \geq k-1}}^n \binom{j-i-1}{k-2} g(s_i, s_j) \right] ds^{(n)}, \end{aligned} \quad (3.20)$$

implies the equality of the right-hand sides of Eqs. (3.18) and (3.19).

Its proof is given in Section 4. It is worth pointing out that the equivalence of Eqs. (3.18) and (3.19) in the special case of GMC follows from the case of $k = 2$ of this identity, which we discussed in [31]. The general case is new and significantly more involved.

We will conclude this section with a brief discussion of higher intermittency derivatives, which one wants to compute to derive the full high temperature (low intermittency) expansion, as we did for the Mellin transform of the total mass of the GMC measure in [23], [26], [31]. It is clear from the structure of the first intermittency derivative in Theorem 3.2 that in order to compute higher derivatives, *i.e.* to apply the differentiation rule iteratively, one needs to establish a differentiation rule for more general functionals of the total mass of the form

$$v(\mu, F, t_1 \cdots t_n) \triangleq \lim_{\varepsilon \rightarrow 0} \mathbf{E} \left[F(M_{\mu, \varepsilon}) e^{\omega_{\mu, \varepsilon}(t_1) + \cdots + \omega_{\mu, \varepsilon}(t_n)} \right], \quad (3.21)$$

which are *non-local*, *i.e.* involve the entire path of dM_μ as opposed to the value of the total mass of the whole interval. We will not attempt to carry out the computation in this paper due to substantial technical challenges associated with it but will indicate two possible approaches. The most direct approach is to replace the limit in Eq. (3.9) with the more general limit

$$\left. \frac{\partial}{\partial \delta} \right|_{\delta=0} \mathbf{E} \left[F(z e^{X(\delta)} M_{\mu, \varepsilon}) (z e^{X(\delta)})^n e^{\omega_{\mu, \varepsilon}(t_1) + \cdots + \omega_{\mu, \varepsilon}(t_n)} \right] \quad (3.22)$$

and repeat the original derivation. The second approach is to interpret the functional in Eq. (3.21) as a change of measure, which is what we did in [21] and [22] in the gaussian case, where the functional in Eq. (3.21) is a simple change of drift. In fact, in the gaussian case one has the identity,

$$\mathbf{E} \left[F(M_{\mu, \varepsilon}) e^{\omega_{\mu, \varepsilon}(t_1) + \cdots + \omega_{\mu, \varepsilon}(t_n)} \right] = \exp \left(\mu \sum_{i < j}^n \rho_\varepsilon(t_j - t_i) \right) \mathbf{E} \left[F \left(\int_0^1 e^{\omega_{\mu, \varepsilon}(s) + \mu \sum_{j=1}^n \rho_\varepsilon(s, t_j)} ds \right) \right], \quad (3.23)$$

which is manifestly non-local. The functional on the right-hand side of Eq. (3.23) is of the form

$$v(\mu, f, F) \triangleq \lim_{\varepsilon \rightarrow 0} \mathbf{E} \left[F \left(\int_0^1 e^{\mu f(s) + \omega_{\mu, \varepsilon}(s)} ds \right) \right], \quad (3.24)$$

where $f(s)$ in our case equals

$$f(s) = \sum_{j=1}^n \rho_\varepsilon(s, t_j). \quad (3.25)$$

The intermittency differentiation rule in the gaussian case for a general $f(s)$ is, cf. [22], [26], and [31],

$$\begin{aligned} \frac{\partial}{\partial \mu} v(\mu, f, F) &= \int_{[0,1]} v(\mu, f + g(\cdot, s), F^{(1)}) e^{\mu f(s)} f(s) ds + \\ &+ \int_{\{s_1 < s_2\}} v(\mu, f + g(\cdot, s_1) + g(\cdot, s_2), F^{(2)}) e^{\mu(f(s_1) + f(s_2) + g(s_1, s_2))} g(s_1, s_2) ds^{(2)}. \end{aligned} \quad (3.26)$$

Hence, applying it to the functional in Eq. (3.21) and using Eq. (3.23), we obtain the desired rule of differentiation that allows one to compute intermittency derivatives of all orders in the gaussian case.

Theorem 3.8 (Gaussian intermittency differentiation)

$$\begin{aligned} \frac{\partial}{\partial \mu} v(\mu, F, t_1 \cdots t_n) &= v(\mu, F, t_1 \cdots t_n) \sum_{i < j}^n g(t_i, t_j) + \int_0^1 v(\mu, F^{(1)}, t_1 \cdots t_{n+1}) \sum_{j=1}^n g(t_j, t_{n+1}) dt_{n+1} + \\ &+ \int_{\{t_{n+1} < t_{n+2}\}} v(\mu, F^{(2)}, t_1 \cdots t_{n+2}) g(t_{n+1}, t_{n+2}) dt_{n+1} dt_{n+2}. \end{aligned} \quad (3.27)$$

We refer the interested reader to [31] for a detailed treatment of the gaussian case. In the general ID case the equivalents of the change of measure in Eq. (3.23) and of the functional in Eq. (3.24) are not known to us and left as open questions.

4 Derivation of the intermittency differentiation rule

In this section we will give derivations of our results.

Proof of Theorem 3.1 The proof is based on Lemma 2.1 and a special property of the function $\rho_\varepsilon(u)$ in Eq. (2.12). First, define

$$\rho_{L,\varepsilon}(u) \triangleq \log L + \rho_\varepsilon(u). \quad (4.1)$$

Then, it is easy to see from Eq. (2.12) that $\rho_{L,\varepsilon}(u)$ satisfies the identity for $|u| < 1$,

$$\delta + \mu \rho_{L,\varepsilon}(u) = (\mu - \delta) \rho_{L,\varepsilon}(u) + \delta \rho_{\varepsilon L,\varepsilon}(u). \quad (4.2)$$

On the other hand, Lemma 2.1 gives us the joint characteristic function of $\omega_{\mu,L,\varepsilon}(t_j)$, $j = 1 \cdots n$, in the form

$$\mathbf{E} \left[\exp \left(i \sum_{j=1}^n q_j \omega_{\mu,L,\varepsilon}(t_j) \right) \right] = \exp \left(\mu \sum_{p=1}^n \sum_{k=1}^p \alpha_{p,k} \rho_{L,\varepsilon}(t_p - t_k) \right), \quad (4.3)$$

where $\rho_{L,\varepsilon}(u)$ is defined in Eq. (4.1) and the coefficients $\alpha_{p,k}$ are the same as in Eq. (2.20). In fact,

$$\begin{aligned} \mathbf{E} \left[\exp \left(i \sum_{j=1}^n q_j \omega_{\mu,L,\varepsilon}(t_j) \right) \right] &= \mathbf{E} \left[\exp \left(i Z_L \sum_{j=1}^n q_j \right) \right] \mathbf{E} \left[\exp \left(i \sum_{j=1}^n q_j \omega_{\mu,\varepsilon}(t_j) \right) \right], \\ &= \exp \left(\mu \phi \left(\sum_{j=1}^n q_j \right) \log L \right) \exp \left(\mu \sum_{p=1}^n \sum_{k=1}^p \alpha_{p,k} \rho_\varepsilon(t_p - t_k) \right), \\ &= \exp \left(\mu \sum_{p=1}^n \sum_{k=1}^p \alpha_{p,k} \log L \right) \exp \left(\mu \sum_{p=1}^n \sum_{k=1}^p \alpha_{p,k} \rho_\varepsilon(t_p - t_k) \right) \end{aligned} \quad (4.4)$$

by Eqs. (2.19) and (2.21) so that Eq. (4.3) now follows from Eq. (4.1). We can now compute the joint characteristic function of the left- and right-hand sides of Eq. (3.4). Using Eqs. (2.21) and (4.3),

$$\mathbf{E} \left[\exp \left(i \sum_{j=1}^n q_j \left(X(\delta) + \omega_{\mu, L, \varepsilon}(t_j) \right) \right) \right] = \exp \left(\sum_{p=1}^n \sum_{k=1}^p \alpha_{p,k} (\delta + \mu \rho_{L, \varepsilon}(t_p - t_k)) \right). \quad (4.5)$$

On the other hand, we have by independence and Eq. (4.3),

$$\begin{aligned} \mathbf{E} \left[\exp \left(i \sum_{j=1}^n q_j \left(\omega_{\mu - \delta, L, \varepsilon}(t_j) + \bar{\omega}_{\delta, eL, \varepsilon}(t_j) \right) \right) \right] &= \exp \left(\sum_{p=1}^n \sum_{k=1}^p \alpha_{p,k} ((\mu - \delta) \rho_{L, \varepsilon}(t_p - t_k) + \right. \\ &\quad \left. + \delta \rho_{eL, \varepsilon}(t_p - t_k)) \right), \end{aligned} \quad (4.6)$$

and the result follows from Eq. (4.2). \blacksquare

The derivation of the intermittency differentiation rule requires three auxiliary lemmas. Recall the definition of the Lévy process in Eq. (3.3) that is associated with the ID distribution specified by its Lévy-Khinchine representation in Eq. (2.2).

Lemma 4.1 (Kolmogorov equation) *Given a test function $v(z)$,*

$$\frac{\partial}{\partial \delta} \Big|_{\delta=0} \mathbf{E} \left[v(z e^{X(\delta)}) \right] = \frac{\sigma^2}{2} z^2 \frac{d^2}{dz^2} v(z) + \int_{\mathbb{R} \setminus \{0\}} \left[v(z e^u) - v(z) - z \frac{d}{dz} v(z) (e^u - 1) \right] d\mathcal{M}(u). \quad (4.7)$$

Proof This is a simple corollary of the backward Kolmogorov equation for the process $X(\delta)$, cf. [7], Section I.2. It is easy to see by following the Fourier-integral type of argument given in [7] that the backward Kolmogorov operator associated with $X(\delta)$,

$$(\mathcal{L}f)(x) \triangleq \frac{\partial}{\partial \delta} \Big|_{\delta=0} \mathbf{E} \left[f(X(\delta) + x) \right], \quad (4.8)$$

is

$$(\mathcal{L}f)(x) = -\frac{\sigma^2}{2} \frac{d}{dx} f(x) + \frac{\sigma^2}{2} \frac{d^2}{dx^2} f(x) + \int_{\mathbb{R} \setminus \{0\}} \left[f(x+u) - f(x) - \frac{d}{dx} f(x) (e^u - 1) \right] d\mathcal{M}(u). \quad (4.9)$$

It remains to apply this formula to $f(x) = v(z e^x)$ at $x = 0$. \blacksquare

Now, recall definitions of $d(m)$ in Eq. (2.22), of the process $\omega_{\mu, L, \varepsilon}(s)$ in Eq. (3.2), and of $\rho_{L, \varepsilon}(u)$ in Eq. (4.1).

Lemma 4.2 (Combinatorial property) *Let $\mathfrak{f}(\delta, s)$ be an arbitrary continuous function that vanishes as $\delta \rightarrow 0$. Let $\mathcal{B}(s) \triangleq e^{\mathfrak{f}(\delta, s) + \omega_{\delta, L, \varepsilon}(s)} - 1$. Then, given any distinct $0 < s_1 < \dots < s_n < 1$, $n \geq 2$, as $\delta \rightarrow 0$,*

$$\mathbf{E} [\mathcal{B}(s_1) \mathcal{B}(s_2)] = (e^{\mathfrak{f}(\delta, s_1)} - 1) (e^{\mathfrak{f}(\delta, s_2)} - 1) + \delta \rho_{L, \varepsilon}(s_2 - s_1) \left(\sigma^2 + \int_{\mathbb{R} \setminus \{0\}} (e^u - 1)^2 d\mathcal{M}(u) \right) + o(\delta), \quad (4.10)$$

$$\mathbf{E} [\mathcal{B}(s_1) \dots \mathcal{B}(s_n)] = (e^{\mathfrak{f}(\delta, s_1)} - 1) \dots (e^{\mathfrak{f}(\delta, s_n)} - 1) + \delta \rho_{L, \varepsilon}(s_n - s_1) \int_{\mathbb{R} \setminus \{0\}} (e^u - 1)^n d\mathcal{M}(u) + o(\delta), \quad (4.11)$$

if $n > 2$.

This lemma generalizes the corresponding result for the gaussian multiplicative chaos measure that we first noted in [21].

Proof We need the following estimate first,

$$\mathbf{E}[\mathcal{B}(s_1) \cdots \mathcal{B}(s_n)] = (e^{\mathfrak{f}(\delta, s_1)} - 1) \cdots (e^{\mathfrak{f}(\delta, s_n)} - 1) + \delta \rho_{L, \varepsilon}(s_n - s_1) \sum_{k=0}^{n-2} (-1)^k \binom{n-2}{k} d_{n-k-1} + o(\delta). \quad (4.12)$$

The proof is essentially based on the identity that follows from Eq. (4.3),

$$\exp\left(\mu \sum_{k < p}^n d(p-k) \rho_{L, \varepsilon}(s_p - s_k)\right) = \mathbf{E}\left[e^{\omega_{\mu, L, \varepsilon}(s_1) + \cdots + \omega_{\mu, L, \varepsilon}(s_n)}\right] \quad (4.13)$$

for any $0 < s_1 < \cdots < s_n < 1$. One then multiplies out the terms on the left-hand side of Eq. (4.12), applies this identity to each resulting term, and differentiates the result with respect to δ . To carry out this calculation in detail, we need to introduce the following notation. Fix $n \geq 2$ and let $(p_k < \cdots < p_k)$ denote a k -tuple of numbers from $\{1 \cdots n\}$. Then, we have the obvious identity that follows from Eq. (4.13) and the vanishing of $\mathfrak{f}(\delta, s)$ as $\delta \rightarrow 0$,

$$\mathbf{E}[\mathcal{B}(s_1) \cdots \mathcal{B}(s_n)] = \prod_{i=1}^n (e^{\mathfrak{f}(\delta, s_i)} - 1) + \delta \sum_{k=2}^n (-1)^{n-k} \sum_{(p_1 < \cdots < p_k)} \sum_{i < j}^k d(j-i) \rho_{L, \varepsilon}(s_{p_j} - s_{p_i}) + o(\delta). \quad (4.14)$$

Now, consider the coefficient of $\rho_{L, \varepsilon}(s_n - s_1)$ first. This means $p_1 = 1$ and $p_k = n$ so that there are

$$\binom{n-2}{k-2} \quad (4.15)$$

such tuples. Hence, the coefficient of $\rho_{L, \varepsilon}(s_n - s_1)$ is

$$\delta \sum_{k=2}^n (-1)^{n-k} \binom{n-2}{k-2} d(k-1), \quad (4.16)$$

which coincides with the expression on the right-hand side of Eq. (4.12) by a change of summation index. Next, consider the coefficient of $d(j-i) \rho_{L, \varepsilon}(s_{p_j} - s_{p_i})$ in general. Let $p_i = a$, $p_j = b$ and $l = j-i$ be fixed. Given, $a < b$ and l , we wish to show that the number of k -tuples that have the property that they contain a and b , i.e. $a = p_i$ and $b = p_j$ for some $i < j$, and $j-i = l$, is

$$\binom{n-1-(b-a)}{k-1-l} \binom{b-a-1}{l-1}. \quad (4.17)$$

For example, let $n = 7$, $k = 4$, $a = 2$, $b = 5$, and $l = 2$. The formula says that there are six such tuples. In fact, they are: (2, 3, 5, 6), (2, 3, 5, 7), (2, 4, 5, 6), (2, 4, 5, 7), (1, 2, 3, 5), (1, 2, 4, 5). To prove the formula, let i be the location of a so the location of b is then necessarily $j = i + l$. The sought number of tuples is then

$$\sum_i \binom{a-1}{i-1} \binom{b-a-1}{l-1} \binom{n-b}{k-j}. \quad (4.18)$$

This formula simply gives us numbers of choices for the elements of the tuple preceding a , in between a and b , and following b . The formula in Eq. (4.17) now follows by the Vandermonde convolution, cf. Eq. (3.1) in [15]. It follows that the coefficient of $d(l)\rho_{L,\varepsilon}(s_b - s_a)$ is

$$\delta \sum_{k=2}^n (-1)^{n-k} \binom{n-1-(b-a)}{k-1-l} \binom{b-a-1}{l-1}. \quad (4.19)$$

It remains to observe that this sum is identically zero provided $n-1 > b-a$, which is a corollary of the classical binomial identity

$$\sum_{k \geq 0} (-1)^k \binom{x}{k} = 0, \quad x > 0, \quad (4.20)$$

cf. Eq. (1.2) in [15]. The case of $n-1 = b-a$ was already treated above, hence Eq. (4.12) is verified. Finally, the alternating sum on the right-hand side of Eq. (4.12) can be simplified using the definition of $d(m)$ in Eq. (2.22) resulting in Eqs. (4.10) and (4.11). ■

Lemma 4.3 (Differentiation) *Let $F(x)$ be a smooth test function. Let*

$$u_\varepsilon(z, \mu, F) \triangleq F\left(z \int_0^1 e^{\omega_{\mu,L,\varepsilon}(s)} ds\right). \quad (4.21)$$

Then, there holds the following identity

$$\frac{\partial}{\partial \mu} u_\varepsilon(z, \mu, F) = - \lim_{\delta \rightarrow 0} \left[\frac{1}{\delta} \sum_{k=1}^{\infty} \frac{u_\varepsilon(z, \mu, F^{(k)})}{k!} \left(z \int_0^1 e^{\omega_{\mu,L,\varepsilon}(s)} (e^{\mathcal{A}_\varepsilon(s)} - 1) ds \right)^k \right], \quad (4.22)$$

where

$$\mathcal{A}_\varepsilon(s) \triangleq \omega_{\mu-\delta,L,\varepsilon}(s) - \omega_{\mu,L,\varepsilon}(s). \quad (4.23)$$

Proof The result follows from writing

$$\int_0^1 e^{\omega_{\mu-\delta,L,\varepsilon}(s)} ds = \int_0^1 e^{\omega_{\mu,L,\varepsilon}(s)} ds + \int_0^1 e^{\omega_{\mu,L,\varepsilon}(s)} (e^{\mathcal{A}_\varepsilon(s)} - 1) ds, \quad (4.24)$$

and Taylor expanding in the “small” parameter

$$\int_0^1 e^{\omega_{\mu,L,\varepsilon}(s)} (e^{\mathcal{A}_\varepsilon(s)} - 1) ds$$

that vanishes as $\delta \rightarrow 0$. ■

We can now give a derivation of Theorem 3.2.

Proof The idea of the derivation is to consider a stochastic flow and derive the corresponding Feynman-Kac equation regarding intermittency as time. Let

$$u_\varepsilon(z, \mu, F) \triangleq F\left(z \int_0^1 e^{\omega_{\mu,1,\varepsilon}(s)} ds\right) \quad (4.25)$$

and let $v_\varepsilon(z, \mu, F)$ be its expectation,

$$v_\varepsilon(z, \mu, F) \triangleq \mathbf{E}[u_\varepsilon(z, \mu, F)]. \quad (4.26)$$

The starting point is the limit

$$A \triangleq \frac{\partial}{\partial \delta} \Big|_{\delta=0} \mathbf{E}^* \left[v_\varepsilon \left(z e^{X(\delta)}, \mu, F \right) \right], \quad (4.27)$$

where $X(\delta)$ is defined by Eq. (3.3) and is independent of $\omega_{\mu,1,\varepsilon}(s)$, and the star is used to distinguish the expectation with respect to $X(\delta)$ from that with respect to $\omega_{\mu,1,\varepsilon}(s)$. By Lemma 4.1, we have

$$A = \frac{\sigma^2}{2} z^2 \frac{\partial^2}{\partial z^2} v_\varepsilon(z, \mu, F) + \int_{\mathbb{R} \setminus \{0\}} \left[v_\varepsilon(z e^u, \mu, F) - v_\varepsilon(z, \mu, F) - z \frac{\partial}{\partial z} v_\varepsilon(z, \mu, F) (e^u - 1) \right] d\mathcal{M}(u). \quad (4.28)$$

On the other hand, this limit can be computed in a different way. By Theorem 3.1, there holds the following equality in law

$$e^{X(\delta)} \int_0^1 e^{\omega_{\mu,1,\varepsilon}(s)} ds = \int_0^1 e^{\omega_{\mu-\delta,1,\varepsilon}(s) + \bar{\omega}_{\delta,\varepsilon,\varepsilon}(s)} ds. \quad (4.29)$$

Thus, to compute the limit in Eq. (4.27), we need to expand

$$\mathbf{E}^* \left[\mathbf{E} \left[F \left(z \int_0^1 e^{\omega_{\mu-\delta,1,\varepsilon}(s) + \bar{\omega}_{\delta,\varepsilon,\varepsilon}(s)} ds \right) \right] \right] - v_\varepsilon(z, \mu, F) \quad (4.30)$$

in δ up to $o(\delta)$ terms. The star now indicates the expectation with respect to $\bar{\omega}_{\delta,\varepsilon,\varepsilon}(s)$, which is independent of $\omega_{\mu-\delta,1,\varepsilon}(s)$ by construction. Let $\mathcal{A}_\varepsilon(s) \triangleq \omega_{\mu-\delta,1,\varepsilon}(s) - \omega_{\mu,1,\varepsilon}(s)$ as in Eq. (4.23) with $L = 1$ and

$$\bar{\mathcal{A}}_\varepsilon(s) \triangleq \bar{\omega}_{\delta,\varepsilon,\varepsilon}(s). \quad (4.31)$$

While we do not know how to expand either $\mathcal{A}_\varepsilon(s)$ or $\bar{\mathcal{A}}_\varepsilon(s)$ in δ , they both clearly vanish as $\delta \rightarrow 0$. It follows that the expression in Eq. (4.30) can be expanded in the “small” quantity

$$\mathcal{C} \triangleq \int_0^1 e^{\omega_{\mu,1,\varepsilon}(s)} (e^{\mathcal{A}_\varepsilon(s) + \bar{\mathcal{A}}_\varepsilon(s)} - 1) ds. \quad (4.32)$$

$$\begin{aligned} \mathbf{E}^* \left[\mathbf{E} \left[F \left(z \int_0^1 e^{\omega_{\mu-\delta,1,\varepsilon}(s) + \bar{\omega}_{\delta,\varepsilon,\varepsilon}(s)} ds \right) \right] \right] &= \mathbf{E}^* \left[\mathbf{E} \left[F \left(z \int_0^1 e^{\omega_{\mu,1,\varepsilon}(s)} ds + z \mathcal{C} \right) \right] \right], \\ &= \mathbf{E}^* \left[\mathbf{E} \left[\sum_{k=0}^{\infty} \frac{z^k}{k!} u_\varepsilon(z, \mu, F^{(k)}) \mathcal{C}^k \right] \right]. \end{aligned} \quad (4.33)$$

The advantage of this representation is that the only $\bar{\omega}_\varepsilon$ dependence is in $\bar{\mathcal{A}}_\varepsilon(s)$. This allows us to compute the \mathbf{E}^* expectation in Eq. (4.30). Indeed, Eq. (4.30) entails two expectations: the \mathbf{E} with respect to ω_ε process inherited from the definition of $v_\varepsilon(z, \mu, F)$ and the \mathbf{E}^* expectation with respect to $\bar{\omega}_\varepsilon$ process. Interchanging their order, it follows from Eq. (4.33) that computing the \mathbf{E}^* expectation is now reduced to computing $\mathbf{E}^*[\mathcal{C}^k]$. As $\mathcal{A}_\varepsilon(s)$ and $\bar{\mathcal{A}}_\varepsilon(s)$ are independent processes, it follows from Lemma 4.2 applied to $\mathcal{B}(s) = \exp(\mathcal{A}_\varepsilon(s) + \bar{\mathcal{A}}_\varepsilon(s)) - 1$ with $\mathfrak{f}(\delta, s) \triangleq \mathcal{A}_\varepsilon(s)$ that the \mathbf{E}^* expectation

equals

$$\mathbf{E}^*[\mathcal{B}(s_1)\mathcal{B}(s_2)] = (e^{\mathcal{A}_\varepsilon(s_1)} - 1)(e^{\mathcal{A}_\varepsilon(s_2)} - 1) + \delta \rho_{e,\varepsilon}(s_2 - s_1) \left(\sigma^2 + \int_{\mathbb{R} \setminus \{0\}} (e^u - 1)^2 d\mathcal{M}(u) \right) + o(\delta), \quad (4.34)$$

$$\mathbf{E}^*[\mathcal{B}(s_1) \cdots \mathcal{B}(s_k)] = (e^{\mathcal{A}_\varepsilon(s_1)} - 1) \cdots (e^{\mathcal{A}_\varepsilon(s_k)} - 1) + \delta \rho_{e,\varepsilon}(s_n - s_1) \int_{\mathbb{R} \setminus \{0\}} (e^u - 1)^k d\mathcal{M}(u) + o(\delta), \quad (4.35)$$

if $k > 2$. Collecting what we have shown so far, we obtain

$$\begin{aligned} A &= \lim_{\delta \rightarrow 0} \frac{1}{\delta} \sum_{k=1}^{\infty} \frac{z^k}{k!} \mathbf{E} \left[u_\varepsilon(z, \mu, F^{(k)}) \left(\int_{[0,1]} e^{\omega_{\mu,1,\varepsilon}(s)} (e^{\mathcal{A}_\varepsilon(s)} - 1) ds \right)^k \right] + \\ &+ \sigma^2 z^2 \int_{\{s_1 < s_2\}} \mathbf{E} \left[u_\varepsilon(z, \mu, F^{(2)}) e^{\omega_{\mu,1,\varepsilon}(s_1) + \omega_{\mu,1,\varepsilon}(s_2)} \right] \rho_{e,\varepsilon}(s_2 - s_1) ds^{(2)} + \\ &+ \sum_{k=2}^{\infty} z^k \int_{\mathbb{R} \setminus \{0\}} (e^u - 1)^k d\mathcal{M}(u) \int_{\{s_1 < \cdots < s_k\}} \mathbf{E} \left[u_\varepsilon(z, \mu, F^{(k)}) e^{\omega_{\mu,1,\varepsilon}(s_1) + \cdots + \omega_{\mu,1,\varepsilon}(s_k)} \right] \rho_{e,\varepsilon}(s_k - s_1) ds^{(k)}. \end{aligned} \quad (4.36)$$

By Lemma 4.3, the $\delta \rightarrow 0$ limit that is involved in Eq. (4.36) equals

$$- \frac{\partial}{\partial \mu} v_\varepsilon(z, \mu, F). \quad (4.37)$$

Observing that $\rho_{e,\varepsilon}(s_2 - s_1) = 1 + \rho_\varepsilon(s_2 - s_1)$ and

$$\begin{aligned} &\int_{\{s_1 < s_2\}} \mathbf{E} \left[u_\varepsilon(z, \mu, F^{(2)}) e^{\omega_{\mu,1,\varepsilon}(s_1) + \omega_{\mu,1,\varepsilon}(s_2)} \right] ds^{(2)} = \frac{1}{2} \frac{\partial^2}{\partial z^2} v_\varepsilon(z, \mu, F), \quad (4.38) \\ &\sum_{k=2}^{\infty} z^k \int_{\mathbb{R} \setminus \{0\}} (e^u - 1)^k d\mathcal{M}(u) \int_{\{s_1 < \cdots < s_k\}} \mathbf{E} \left[u_\varepsilon(z, \mu, F^{(k)}) e^{\omega_{\mu,1,\varepsilon}(s_1) + \cdots + \omega_{\mu,1,\varepsilon}(s_k)} \right] ds^{(k)} = \\ &= \int_{\mathbb{R} \setminus \{0\}} \left[v_\varepsilon(ze^u, \mu, F) - v_\varepsilon(z, \mu, F) - z \frac{\partial}{\partial z} v_\varepsilon(z, \mu, F) (e^u - 1) \right] d\mathcal{M}(u), \quad (4.39) \end{aligned}$$

and substituting these equations into Eq. (4.36), we obtain

$$\begin{aligned} A &= - \frac{\partial}{\partial \mu} v_\varepsilon(z, \mu, F) + \frac{\sigma^2 z^2}{2} \frac{\partial^2}{\partial z^2} v_\varepsilon(z, \mu, F) + \\ &+ \int_{\mathbb{R} \setminus \{0\}} \left[v_\varepsilon(ze^u, \mu, F) - v_\varepsilon(z, \mu, F) - z \frac{\partial}{\partial z} v_\varepsilon(z, \mu, F) (e^u - 1) \right] d\mathcal{M}(u) + \\ &+ \sigma^2 z^2 \mathbf{E} \int_{\{s_1 < s_2\}} \left[u_\varepsilon(z, \mu, F^{(2)}) e^{\omega_{\mu,1,\varepsilon}(s_1) + \omega_{\mu,1,\varepsilon}(s_2)} \right] \rho_\varepsilon(s_2 - s_1) ds^{(2)} + \\ &+ \sum_{k=2}^{\infty} z^k \int_{\mathbb{R} \setminus \{0\}} (e^u - 1)^k d\mathcal{M}(u) \int_{\{s_1 < \cdots < s_k\}} \mathbf{E} \left[u_\varepsilon(z, \mu, F^{(k)}) e^{\omega_{\mu,1,\varepsilon}(s_1) + \cdots + \omega_{\mu,1,\varepsilon}(s_k)} \right] \rho_\varepsilon(s_k - s_1) ds^{(k)}. \end{aligned} \quad (4.40)$$

Finally, upon comparing this expression for A with that in Eq. (4.28), and then letting $z = 1$ and $\varepsilon \rightarrow 0$, we arrive at Eq. (3.8). \blacksquare

Proof of Theorem 3.4 The proof is very similar to that of Theorem 3.2 so it is sufficient to point out that instead of the limit in Eq. (4.27) one needs to evaluate the more general limit

$$A \triangleq \frac{\partial}{\partial \delta} \Big|_{\delta=0} \mathbf{E} \left[F_1 \left(z e^{X(\delta)} \int_{I_1} e^{\omega_{\mu,1,\varepsilon}(s)} ds \right) F_2 \left(z e^{X(\delta)} \int_{I_2} e^{\omega_{\mu,1,\varepsilon}(s)} ds \right) \right], \quad (4.41)$$

The remaining details are essentially the same and will be omitted. \blacksquare

Proof of Corollary 3.6 The argument is based on Corollary 3.5. Let

$$F_1(x) = F_2(x) = \log x. \quad (4.42)$$

The intervals I_1 and I_2 are

$$I_1 = [0, \tau], \quad I_2 = [t, t + \tau], \quad (4.43)$$

and the expansion is around the point $x = |I_1| = |I_2| = \tau$ so that the necessary derivative is

$$F^{(k)}(\tau) = (-1)^{k-1} (k-1)! \tau^{-k}. \quad (4.44)$$

Using the standard formula for the covariance of two random variables

$$\mathbf{Cov}(A, B) = \mathbf{E}[AB] - \mathbf{E}[A]\mathbf{E}[B], \quad (4.45)$$

and applying Corollary 3.5 in the limit $\tau \rightarrow 0$ it is not difficult to see that $\log \tau$ terms cancel out and the remaining terms are

$$\begin{aligned} \mathbf{Cov} \left(\log \int_t^{t+\tau} M_\mu(dt), \log \int_0^\tau M_\mu(dt) \right) &= \mu \left[\frac{\sigma^2}{\tau^2} \int_{\{s_1 \in I_1, s_2 \in I_2\}} g(s_1, s_2) ds^{(2)} + \right. \\ &\quad + \sum_{\substack{k,l \geq 0 \\ k+l \geq 2}}^{\infty} (-1)^{k+l} (k-1)! (l-1)! \tau^{-(k+l)} \times \\ &\quad \times \int_{\mathbb{R} \setminus \{0\}} (e^u - 1)^{k+l} d\mathcal{M}(u) \int_{\substack{\{s_1 < \dots < s_k\} \subset I_1 \\ \{s_{k+1} < \dots < s_{k+l}\} \subset I_2}} g(s_1, s_{k+l}) ds^{(k+l)} \Big] + o(1). \end{aligned} \quad (4.46)$$

Finally, in the limit $\tau \rightarrow 0$ the integrals can be obviously approximated by

$$\int_{\substack{\{s_1 < \dots < s_k\} \subset I_1 \\ \{s_{k+1} < \dots < s_{k+l}\} \subset I_2}} g(s_1, s_{k+l}) ds^{(k+l)} = \frac{\tau^{k+l}}{k!l!} g(t) + O(\tau), \quad (4.47)$$

and the result follows from the power series expansion of the logarithm function. \blacksquare

Proof of Theorem 3.7 We first need to simplify the sum in Eq. (3.18). The starting point is the identity

$$\sum_{k < p}^n d(p-k) g(t_p, t_k) = \sigma^2 \sum_{k < p}^n g(t_p, t_k) + \sum_{k=2}^n \int_{\mathbb{R} \setminus \{0\}} (e^u - 1)^k d\mathcal{M}(u) \left[\sum_{\substack{i < j \\ j-i \geq k-1}}^n \binom{j-i-1}{k-2} g(t_i, t_j) \right]. \quad (4.48)$$

Its proof is a simple application of Eq. (2.22), which implies

$$\sum_{k < p}^n d(p-k) g(t_p, t_k) = \sigma^2 \sum_{k < p}^n g(t_p, t_k) + \int_{\mathbb{R} \setminus \{0\}} (e^u - 1)^2 d\mathcal{M}(u) \left[\sum_{k < p}^n e^{(p-k-1)u} g(t_k, t_p) \right]. \quad (4.49)$$

Now, using the identity

$$e^{pu} = \sum_{s=0}^p (e^u - 1)^s \binom{p}{s}, \quad (4.50)$$

and changing the order of summation, we obtain

$$\sum_{k < p}^n d(p-k) g(t_p, t_k) = \sigma^2 \sum_{k < p}^n g(t_p, t_k) + \int_{\mathbb{R} \setminus \{0\}} d\mathcal{M}(u) \left[\sum_{s=0}^{n-2} (e^u - 1)^{s+2} \sum_{\substack{k < p \\ p-k-1 \geq s}}^n \binom{p-k-1}{s} g(t_k, t_p) \right] \quad (4.51)$$

which is equivalent to Eq. (4.48).

We note next that the product in Eq. (3.18) satisfies

$$\prod_{k < p}^n r(t_p - t_k)^{-\mu d(p-k)} = \lim_{\varepsilon \rightarrow 0} \mathbf{E} \left[e^{\omega_{\mu, \varepsilon}(t_1) + \dots + \omega_{\mu, \varepsilon}(t_n)} \right] \quad (4.52)$$

for any $0 < t_1 < \dots < t_n < 1$, cf. Eq. (2.24). Then, upon substituting this equation into Eq. (3.18), we see that to establish the equivalence of Eqs. (3.18) and (3.19) it is sufficient to verify the identity in Eq. (3.20).

Before we give a formal proof of Eq. (3.20), we will treat the special case of $n-k=1$ to illustrate the main idea. By breaking up the integration region of the ds integral into three subregions, we can write

$$\begin{aligned} \int_0^1 e^{\omega(s)} ds \int_{\{s_1 < \dots < s_k\}} e^{\omega(s_1) + \dots + \omega(s_k)} g(s_1, s_k) ds^{(k)} &= \int_{\{s < s_1 < \dots < s_k\}} e^{\omega(s) + \omega(s_1) + \dots + \omega(s_k)} g(s_1, s_k) ds^{(k+1)} + \\ &+ \sum_{i=1}^{k-1} \int_{\{s_1 < \dots < s_i < s < s_{i+1} < \dots < s_k\}} e^{\omega(s) + \omega(s_1) + \dots + \omega(s_k)} g(s_1, s_k) ds^{(k+1)} + \\ &+ \int_{\{s_1 < \dots < s_k < s\}} e^{\omega(s) + \omega(s_1) + \dots + \omega(s_k)} g(s_1, s_k) ds^{(k+1)}. \end{aligned} \quad (4.53)$$

We now relabel the variables of integration, resulting in the identity

$$\begin{aligned} \int_0^1 e^{\omega(s)} ds \int_{\{s_1 < \dots < s_k\}} e^{\omega(s_1) + \dots + \omega(s_k)} g(s_1, s_k) ds^{(k)} &= \int_{\{s_1 < \dots < s_{k+1}\}} e^{\omega(s_1) + \dots + \omega(s_{k+1})} \left[g(s_2, s_{k+1}) + \right. \\ &\left. + (k-1)g(s_1, s_{k+1}) + g(s_1, s_k) \right] ds^{(k+1)}, \end{aligned} \quad (4.54)$$

which is the same as the expression on the right-hand side of Eq. (3.20). To prove Eq. (3.20) in general, it is clear that one can apply the above procedure iteratively resulting in an identity of the

form

$$\frac{1}{(n-k)!} \left(\int_0^1 e^{\omega(s)} ds \right)^{n-k} \int_{\{s_1 < \dots < s_k\}} e^{\omega(s_1) + \dots + \omega(s_k)} g(s_1, s_k) ds^{(k)} = \int_{\{s_1 < \dots < s_n\}} e^{\omega(s_1) + \dots + \omega(s_n)} \times \left[\sum_{i < j}^n C_{ijk} g(s_i, s_j) \right] ds^{(n)} \quad (4.55)$$

with some coefficients C_{ijk} to be determined. Now, the left-hand side can be obviously written as

$$\frac{1}{(n-k)!} \left(\int_0^1 e^{\omega(s)} ds \right)^{n-k} \int_{\{s_1 < \dots < s_k\}} e^{\omega(s_1) + \dots + \omega(s_k)} g(s_1, s_k) ds^{(k)} = \int_{\{t_1 < \dots < t_{n-k}\}} e^{\omega(t_1) + \dots + \omega(t_{n-k})} dt^{(n-k)} \int_{\{s_1 < \dots < s_k\}} e^{\omega(s_1) + \dots + \omega(s_k)} g(s_1, s_k) ds^{(k)}. \quad (4.56)$$

Then, following the logic of the calculation in the case of $n-k=1$ above, the coefficient C_{ijk} equals the number of ways of inserting the t 's into $s_1 < \dots < s_k$ so that s_1 ends up in the position i and s_k in the position j after the insertion. This means that there should be $j-i-1$ variables in between s_1 and s_k , of which there are $k-2$ s 's and $j-i-1-(k-2)$ t 's. Thus, C_{ijk} equals the number of ways of choosing $k-2$ locations out of $j-i-1$ available positions, *i.e.*

$$C_{ijk} = \binom{j-i-1}{k-2}. \quad (4.57)$$

This completes the proof. ■

5 Conclusions

We have presented a theory of intermittency differentiation for a general class of 1D infinitely divisible multiplicative chaos measures on the interval including the canonical measures of Bacry-Muzy as a special case. The rule of intermittency differentiation is an exact, non-local, Feynman-Kac equation that expresses the intermittency derivative of the expectation of a test function of the total mass on one or more subintervals of the unit interval in terms of the derivatives of the function and the Lévy-Khinchine formula of the underlying distribution. The equation is based on the intermittency invariance of the underlying infinitely divisible field that we established in full generality in this paper. This invariance is a novel technical device that substitutes for the non-existent Markov property of the underlying field and allows one to derive a Feynman-Kac equation for the distribution of the total mass by considering a stochastic flow in intermittency (as opposed to time in the classical framework of diffusions). The intermittency invariance gives two ways of evaluating the limit of the flow, which results in the differentiation rule. The first way is the backward Kolmogorov equation for Lévy processes, the second way involves detailed analysis of certain infinite series expansions, combined with a key combinatorial property of the measure that we derived in the paper. Our analysis of these expansions is exact at the level of formal power series but not mathematically rigorous as we have not examined their convergence properties.

We illustrated the rule of intermittency differentiation with two examples. The first is that of positive integer moments of the total mass of the measure. Given the known multiple integral representation of these moments, we rigorously proved that the intermittency differentiation rule for the

moments coincides with the formula for the intermittency derivative that follows from the multiple integral representation. In the second example we derived a formula for the covariance of the total mass from the differentiation rule for two subintervals.

The intermittency differentiation rule is the first step towards a perturbative expansion of the distribution of the total mass and, more generally, the dependence structure of the limit measure in powers of intermittency, *i.e.* the high temperature expansion. We limited ourselves in this paper to the differentiation rule for the first derivative and evaluated it explicitly. Our approach can be naturally extended to higher derivatives by applying our technique to a class of non-local functionals of the total mass that we introduced in the paper. To illustrate this point, we stated the general differentiation rule that allows one to compute all higher order derivatives in the special case of Gaussian Multiplicative Chaos. The actual computation of higher order derivatives in the infinitely divisible case is technically more difficult and left to future research as is the problem of renormalizability of the full intermittency expansion. Finally, we also want to mention the very interesting open problem of computing positive integer moments explicitly, *i.e.* computing the generalized Selberg integral corresponding to the underlying infinitely divisible distribution and intensity measure that is defined in the paper.

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